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REMARKS ON THE MATHEMATICAL THEORY OF DETONATION AND DEFLAGRATION WAVES IN GASES

(Supplement to the Manual on Supersonic Flow and Shock Waves)

Prepared for the
APPLIED MATHEMATICS PANEL
NATIONAL DEFENSE RESEARCH COMMITTEE
By the
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Approved for Distribution
Warren Weaver
Chief, Applied Mathematics Panel, NDRC

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PREFACE

In the present supplement to the Manual on Supersonic Flow and Shock Waves the gas dynamical phenomena of one-dimensional flow involving detonation and combustion are analyzed from the mathematician's viewpoint. As in the Manual, content and emphasis in the present supplement are conditioned by the background from which the writers happened to approach the subject; important points, such as the finite width of the reaction zone, are touched upon only in an appendix and in the bibliography.

Jouguet, in his classical work, was concerned mainly with the discussion of the discontinuous reaction front; but only the consideration of the flow as a whole can supply the information necessary to determine the dynamical phenomena involving detonation or combustion. G. I. Taylor has studied detonation processes under such aspects. In the recent research program of the Applied Mathematics Group of New York University a somewhat more systematic analysis of the mathematical possibilities of flows involving reaction processes became desirable, and the present report gives an account of such investigations.

R. Courant
Director, Contract OEMsr-945

REMARKS ON THE MATHEMATICAL THEORY OF DETONATION AND DEFLAGRATION WAVES IN GASES

(Supplement to the Manual on Supersonic Flow and Shock Waves)

AMP Report 38.3R

In a process of detonation as well as of combustion (or deflagration) a "reaction front" sweeps over the combustible substance, here assumed as a gas, and separates the explosive from the burnt gases. As seen in the "Manual on Supersonic Flow and Shock Waves" (AMP Report 38.2R) there is a close analogy between "shock fronts" and "reaction fronts," both are discontinuity surfaces across which the state of the substance undergoes sudden changes by the three laws of mass, of momentum, and of energy; the difference is merely that across reaction fronts the energy-balance contains as one term the energy liberated by the reaction, while no such term appears in the energy condition for shocks.*

Together with the differential equations in the zones of continuity and with initial and boundary conditions, the transition conditions should determine the flow altogether. However, while such a determinancy exists for gas motions involving shocks only, a different situation presents itself as soon as "reaction fronts" occur (we shall use this term indiscriminately for the transition front due to detonation or deflagration); then the conditions as usually envisaged in non-reactive gas flows are no longer sufficient to determine the flow, and additional conditions must be found; in the case of deflagration or combustion the degree of indeterminacy is even higher than for detonation fronts.

* See Manual [1], Chapter III, Section 50.

Thus the following general problem arises: To what extent are reaction waves left undetermined by the laws of conservation? What are appropriate additional conditions to determine the reaction front? These questions are of a strictly mathematical character. They refer to the determinacy of the solutions of the underlying differential equations satisfying well-defined transition, initial, and boundary conditions.

Implicitly the assumption seems to have always been made that for detonation waves the well known Chapman-Jouguet rule as an additional condition suffices to determine the process while for combustion fronts other conditions have to be supplied from a specific theory of the mechanism of combustion. Our following mathematical analysis has the objective of clarifying this statement and of specifying which reaction fronts are compatible with conditions as they actually may occur. Moreover, it will be seen in what manner reaction fronts can and must be accompanied by rarefaction waves, compression waves or shocks. The mathematical analysis, restricted here to the case of one-dimensional motion <u>in gases</u>, will be based on the theory of characteristics; in particular, on the concept of domain of dependence.*

While the following discussion stresses the mathematical aspect of the theory, it may have physical significance in particular for the understanding of reaction processes taking place in gases which are in non-uniform motion.

<u>Remarks on the mathematical theory of detonation and deflagration waves in gases.</u>

1. <u>Basic notions concerning reaction fronts.</u>**

We first recall a number of well-known facts concerning reaction fronts in gases. The <u>gases</u>, before and after reaction are assumed <u>ideal</u> but not necessarily polytropic. In other words,

* See Courant-Hilbert [2], Chapter 5, and the Manual [1], Chapt II.

** For the physico-chemical notions and facts in the present report Lewis and v. Elbe [3]. For the theory of reaction fronts see Jouguet [4],[5], Becker [6], v. Neumann [7],Taylor [14].

the internal energy is a function of the temperature but not necessarily proportional to it. By p, ρ, $\tau = \rho^{-1}$, c we denote pressure, density, specific volume, and sound speed of the burnt gas quantities referring to the unburnt gas or "explosive" are characterized by a subscript (o). We introduce the quantity

$$\Theta = \tau p,$$

which is proportional* to the temperature T. By e, $i = e + \Theta$, g we denote respectively the internal energy, enthalpy, and energy of formation** per unit mass. The quantities

$$E = e + g, \quad I = i + g$$

may conveniently be called "chemical energy" and "chemical enthalpy" p.u. mass respectively. We assume that these quantities depend only on the temperature and hence*** on Θ.

The sound speed c is defined by the relation $dp - c^2 d\rho = 0$ if $dE + p d\tau = 0$; the latter relation expresses that p and τ are coupled adiabatically. We introduce a function $\gamma = \gamma(\Theta)$ of Θ by the relation

(1) $$\frac{dE}{d\Theta} = \frac{1}{\gamma - 1} .$$

---

* $\Theta = M^{-1}RT$, R being the gas constant, M the molecular weight. Since the composition of the burnt gas and hence its molecular weight depends on the temperature, the dependence of Θ on T is not strictly linear.

** With reference to absolute zero temperature. By liberated energy f we shall denote the difference $g_o - g$ of the energies of formation at absolute zero temperature (not, as more customary, at a standard temperature such as 298^oK.)

*** Strictly speaking this is not correct for the burnt gas, which at different temperatures has a different chemical composition, since the state of equilibrium depends not only on the temperature but also to a certain degree on the pressure. In the following we ignore this dependence.

Then (see App. II) the sound speed is given by*

(2) $$c^2 = \gamma\theta .$$

We assume the gas flow to be one-dimensional, taking place in a long cylindrical tube. By u, $\dot{\xi}$, and

(3) $$v = u - \dot{\xi}$$

we denote gas velocity, reaction front velocity, and relative velocity respectively.

The laws of conservation in the regions where the flow is continuous and adiabatic are expressed by the differential equations

$$\rho_t + (u\rho)_x = 0 ,$$

$$u_t + uu_x + \tau p_x = 0 ,$$

$$E_t + uE_x + p(\tau_t + u\tau_x) = 0 .$$

Across a discontinuity front the laws of conservation are

I $$\rho v = \rho_o v_o ,$$

II $$p + \rho v^2 = p_o + \rho_o v_o^2 ,$$

III $$\frac{1}{2}v^2 + I = \frac{1}{2}v_o^2 + I_o .$$

---

* For we have identically

$$dE + pd\tau = \frac{1}{\gamma - 1}(\tau dp + \gamma pd\tau) = \frac{\tau}{\gamma - 1}(dp - \gamma\theta d\rho),$$

hence vanishing of $dE + pd\tau$ implies $dp/d\rho = \gamma\theta$.

The mechanical relations I and II imply relations

$$\frac{p-p_o}{\tau_o-\tau} = \rho^2 v^2 = \rho_o^2 v_o^2 \quad , \tag{4}$$

$$\frac{p-p_o}{\rho-\rho_o} = v_o v \quad , \tag{5}$$

$$(p-p_o)(\tau_o - \tau) = (v-v_o)^2 \quad , \tag{6}$$

$$\frac{(p-p_o)}{u_o-u} = \rho v = \rho_o v_o \quad , \tag{7}$$

to which we shall refer frequently.

To analyze the significance of the thermodynamical condition III we eliminate v and v_o and obtain

$$I^{(\theta)} - I_o = \frac{1}{2}(\tau + \tau_o)(p - p_o) \quad \text{or} \tag{IV'}$$

$$E^{(\theta)} - E_o = \frac{1}{2}(\tau - \tau_o)(p + p_o) \quad . \tag{IV}$$

Equation IV represents a relation between τ and p, when ρ_o, p_o, and hence θ_o are kept fixed. Its graph in a (τ, p)-plane is indicated in the diagram. For polytropic gases with γ = const., the graph is evidently a hyperbola, since then E is proportional to $p\tau$, as in I.

As will be seen later, (see Fig.4), it is only to the branches $\tau \leq \tau_o$ and to the branch $p \leq p_o$ that real values of the velocities v_o, v correspond. Every point on these two branches corresponds to a possible state of the burnt gas after it has

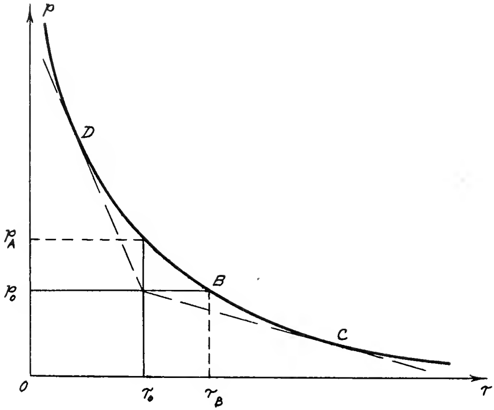

Figure 1
Relation between pressure and specific volume behind a reaction front.

undergone a transition through a reaction front from a state (τ_0, p_0) which characterizes the unexploded gas. A transition to a state characterized by a point on the branch $\tau \leq \tau_0$ is customarily identified with a detonation. The branch $p \leq p_0$ corresponds to deflagration or combustion. The transition to p_A, $\tau_A = \tau_0$, $E_A = E_0$ corresponds to a "constant volume detonation"; (in this limiting case the velocity v_0 is infinite) the transition to $p_B = p_0$, τ_B, $I_B = I_0$ corresponds to a constant pressure combustion (here the velocities v and v_0 are zero).* In general, however, a detonation implies decrease of specific volume (or increase of density), while combustion implies decrease of pressure.

We assume the nature of the gases is such that $p_A > p_0$; in other words, that a constant volume detonation raises the

* The proof is implied in relation (4).

pressure (cf. Becker [6]). As a consequence we have also $\tau_B > \tau_o$; in other words, a constant pressure deflagration raises the specific volume. We further assume that $dp/d\tau < 0$,* $d^2p/d\tau^2 > 0$.

To exhibit how the gas velocities behave behind a reaction front we represent the transitions by a graph, representing for fixed u_o, p_o, τ_o in a (u,p)-plane the relation between p and the velocity jump $u - u_o = v - v_o$. This relation is obtained by elimination of τ from relations IV and (6). The graph in the (u,p)-plane is indicated in Fig. 2.

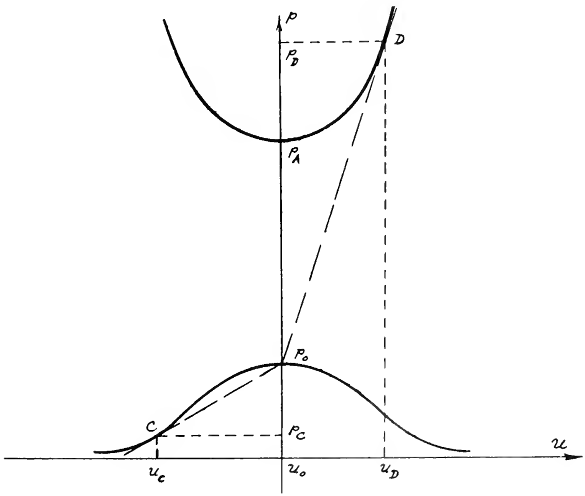

Figure 2
Relation between pressure and velocity behind a reaction front.

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*Since by IV, with $\mu^2 = (\gamma - 1)/(\gamma + 1)$, $\frac{dp}{d\tau} = -\frac{p + \mu^2 p_o}{\tau - \mu^2 \tau_o}$, condition $dp/d\tau < 0$ is equivalent with $\tau > \mu^2 \tau_o$ throughout, which we shall assume. Note that for most gases $\gamma \rightarrow 1$ and hence $\mu^2 \rightarrow 0$ as $\theta \rightarrow \infty$.

A transition to a state represented by a point on the upper branch corresponds to a detonation, while the lower branch corresponds to a deflagration. If the reaction front is "facing to the right," i.e. if the explosive gas (0) is to the right and the burnt gas to the left, (the positive x-axis defining the direction to the right), the mass flux is from the right to the left, $v_o < 0$, and consequently

$$\frac{p-p_o}{u-u_o} = -\rho_o v_o > 0.$$

Therefore, the right upper and the left lower branch correspond to forward facing reactions. It is then clear that for a forward facing detonation $u > u_o$, while for a forward facing deflagration $u < u_o$. In other words:

<u>A detonation accelerates the burnt gas toward the explosive, a deflagration accelerates the burnt gas away from the explosive.</u>

Among all conceivable reaction processes which lead from a fixed state (p_o, u_o, τ_o) to states $(p, u, \tau,)$, there are two of particular significance, one a deflagration and one a detonation. They are characterized by the "Chapman-Jouguet law"* which can be stated in various equivalent forms; for most reaction processes it becomes plausible by many considerations (see e.g. subsequent discussions in this supplement), but must be considered as a hypothesis rather than a law of general validity. The "Chapman-Jouguet reactions" can be defined by any of the following conditions P, Q, R, S, T:

$$\text{(P)} \qquad \frac{dp}{d\tau} = \frac{p-p_o}{\tau-\tau_o}$$

(That there are just two solutions of this equation, one with $p > p_o$ and one with $\tau > \tau_o$, follows from the assumptions $d^2p/d\tau^2 > 0$, $\frac{dp}{d\tau} < 0$, and $p_o < p_A$ or $\tau_o < \tau_B$, see p. 10.) On the graph of p as a function of τ (Fig. 1) the point corresponding

* See Manual [1].

to this transition is characterized by the condition that the tangent at this point passes through the point (τ_o, p_o). Similarly, on the graph in the (u,p)-plane (Fig.2) this point is characterized by the condition that the tangent at this point passes through (u_o, p_o)*. Equivalent properties are:** (See Manual p. 131)

(Q) The propagation speed $|v_o|$ of the front observed from the explosive is an extremum for fixed values of u_o, p_o, τ_o) (Chapman 1899), or

$$dv_o = 0 .$$

(R) The entropy of the burnt gas is an extremum.

(S) The propagation speed $|v|$ of the front observed from the burnt gas is sonic (Jouguet 1905)

$$|v| = c .$$

(T) $$\frac{dp}{d\tau} = -\gamma\frac{p}{\tau} .$$

The two Chapman-Jouguet reaction processes, corresponding to the points D and C, differ in that <u>for detonations the speed $|v_o|$ and the entropy are relative minima</u>, while for <u>deflagrations the speed $|v_o|$ and the entropy are relative maxima</u>, the state (O) ahead of the wave always assumed to be fixed (see App.I.3).***

---

* From (4) we have by (P)
$2(u-u_o)du = (\tau_o-\tau)dp - cp - p_o)d\tau = 2(\tau_o-\tau)dp$, whence
$du/dp = (u-u_o)/(p-p_o)$.

** (For the equivalence of the conditions (P) to (T) (see App.I.1). A procedure for computing the Chapman-Jouguet reaction process when the state in front of it is given is outlined in Appendix II.

*** Emphasis is laid on the relative character of these extrema. The absolute value of the maximum for deflagrations may be less than that of the minimum for detonations.

the extremal properties of the speed $|v_o|$ can be understood* from the relation (4),

$$\rho_o^2 v_o^2 = \frac{p-p_o}{\tau_o-\tau},$$

and the fact that $\frac{p-p_o}{\tau-\tau_o}$ is the slope of the segment connecting the points (τ_o,p_o) and (τ,p) together with $d^2p/d\tau^2 > 0$.

According to whether or not a discontinuity effects a smaller or larger change, a detonation implying a transition of p_o to $p < p_D$ or $p > p_D$ will be called weak or strong respectively; a deflagration implying a transition of p to $p > p_C$ or $p < p_C$ will also be called weak or strong respectively. Both a constant volume detonation and a constant pressure deflagration are weak.

The answer to our main question, the problem of determinacy of reaction fronts, will depend decisively on whether the flow, observed from the reaction front, is sonic or subsonic or supersonic. This question is answered by a statement (Jouguet [5](which we shall prove in Appendix I.3), and which will serve as the basis of the subsequent analysis:
The gas flow relative to the reaction front is

supersonic ahead of a detonation front,
supersonic behind a weak detonation front,
subsonic behind a strong detonation front,
subsonic ahead of a deflagration front,
subsonic behind a weak deflagration front,
supersonic behind a strong deflagration front.

We shall refer to and make use of these properties as Jouguet's rule in the following discussion.

---

* One might also use the relation (7)

$$v_o = -\tau_o \frac{p-p_o}{u-u_o},$$

and the fact that $\frac{p-p_o}{u-u_o}$ is the slope of the segment connecting the points (u_o,p_o) and (u,p). This fact leads to an obvious geometric construction of v_o in the (u,p)-plane, see Courant-Friedrichs [8], p. 51.

2. <u>General remarks on the determinacy of a flow involving a discontinuity front.</u>

As was said before, we restrict our discussion to the case of gases filling a long tube and assume that all processes are one-dimensional, (an assumption which will be actually satisfied for combustion processes only if the tube has a considerable width).

Assuming that the state of the gas at some initial time is given and that at this initial time a reaction front occurs at some place, we ask how far the flow is then determined by the three laws of conservation and the boundary conditions imposed at the ends of the tube. (These conservation laws are, as said before, nothing but the flow differential equations in the continuous regions of the flow and the transition relations I, II, III at a reaction front.)

We specify our initial conditions by assuming that at time $t = 0$, the explosive is at rest in the half infinite tube $x \geq 0$, that a reaction begins at $t = 0$ at the closed end $x = 0$, and that the closed end is operated as a piston moving with the prescribed velocity U.

Usually one would be concerned with the case $U = 0$, which means that the end of the tube is held fixed; however, considering the general case of an arbitrary value U not only contributes much to the clarification of the problem of determinacy, but also is helpful for the understanding of phenomena such as occur when an explosive gas moves against a wall and is ignited there; likewise for an understanding of the phenomena of boosting, the study of the case $U \neq 0$ may be useful.

The mathematical determinacy of flows involving reaction fronts depends essentially on the geometrical relationship between the characteristics of the system of differential equations I, II, III, and the line W representing the reaction front in the (x,t) plane as well as the line π representing the moving piston at the closed end of the tube. As usual, the one dimensional

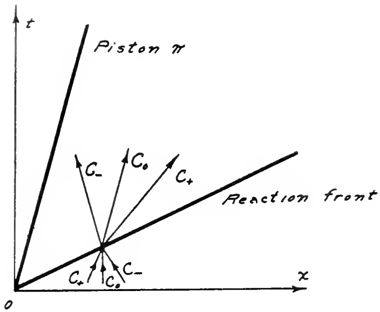


Figure 3
Possible relation between characteristic and reaction front. (For other possibilities see Fig.7.)

motion is represented in the (x,t) -plane. For isentropic flow the system reduces to the system of the differential equation I and II only (see Manual [1], Chapter III) and for any flow there exists two sets of characteristics C_-, C_+ in the (x,t)-plane, defined (see Manual, Chapter IV) by

$$C_+ : dx = (u+c)dt; \text{ along } C_+ \text{ we have } dp + \rho c\, du = 0$$

$$C_- : dx = (u-c)dt; \text{ along } C_- \text{ we have } dp - \rho c\, du = 0$$

If the flow is not isentropic the third family of characteristics C_o, the streamlines, have to be considered

$$C_o + dx = udt; \text{ along } C_o \text{ we have } d\,E + pdt = 0$$

For the following discussion we envisage the case of isentropic flow, keeping in mind that the situation is analogous in the general case. Furthermore, if nothing is said to the contrary, we shall visualize the flow in front and behind the reaction front as consisting of zones of constant flow and simple waves, so that either both sets C_- and C_+ or at least one is straight. (The following statements, however, remain valid also for flows

in which the characteristics are all curved). We imagine the characteristics described in the direction of increasing values of t. Then a line element λ through a point P in the (x,t) -plane is called <u>space-like</u> if the two characteristics leading into P are both on the same side of λ ; otherwise is called <u>time-like</u> (see Fig. 3).

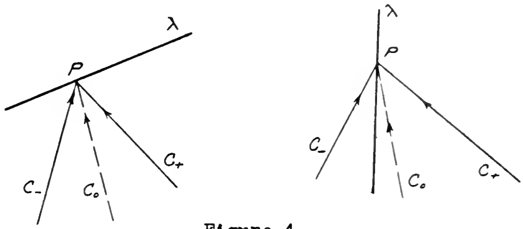


Figure 4

space-like time-like

A line or curve through a zone of flow in the (x,t) - plane is called space-like if all its line elements are space-like. (E.g. t = 0, the x-axis is space-like.)

If initial values of u and ρ are arbitrarily prescribed along a space-like curve L (see Fig. 5), then the initial value

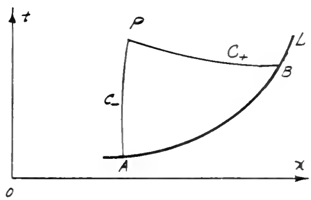


Figure 5
Domain of dependence,
AB, of a point P.

problem is uniquely solvable, and the values of u and ρ at a point P, as in Fig. 5, depend only on the initial values of u and ρ on that part AB of L which is cut out by the two characteristics drawn backwards from L domain of dependence).

If L were time-like, then only one of the two characteristics would intersect L; along a time-like line not both quantities u and ρ can be prescribed arbitrarily, but only one of them. The same is true if L is characteristic.

If u and ρ are prescribed on the space-like initial curve L: AB then the flow is determined between L and the characteristic C_-^o through A. To determine the flow beyond C_-^o further conditions must be given. In particular, let us consider the sector G between C_-^o and a time-like curve J, then the existence of one solution is uniquely determined if along G only one of the quantities u or ρ is (arbitrarily) prescribed.

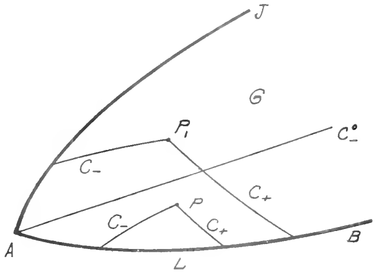


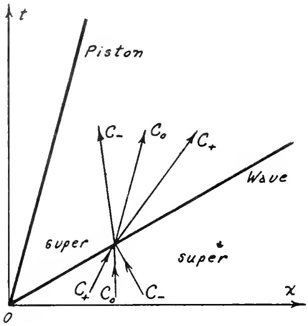
Figure 6
Along space-like L two quantities u, ρ can be prescribed; along time-like J only one quantity.

Generally speaking: The number of conditions to be prescribed along a portion of the boundary of the domain in the (x,t) - plane is given by the number of intersections with characteristics drawn backwards from points near the considered portion of the boundary.

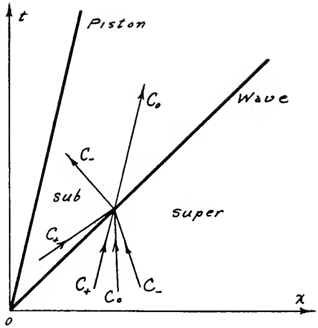
> In the case of a non-isentropic flow, also the third set of characteristics C_0, the streamlines, has to be considered. But the situation remains similar. We draw from a point P the characteristics backwards (towards decreasing values of t); then along a space-like part of the boundary of a region g in the (x,t) - plane three unknown quantities, u, ρ and say the pressure p, can be prescribed. The number of quantities prescribable along a time-like line T is one if only one of the characteristics, say C_+, through points of $\mathcal{J}$ leads into g, while two quantities may be prescribed, when there are two such characteristics, say C_+ and C_0.

After these remarks the various possibilities of flows involving reaction fronts can easily be analyzed. The domains G in which we are primarily interested are first that between the (space-like) x-axis and the <u>reaction line</u> W representing the front in the (x,t)- plane, and second the region between the reaction line W and the <u>piston curve</u> π: x = Ut.

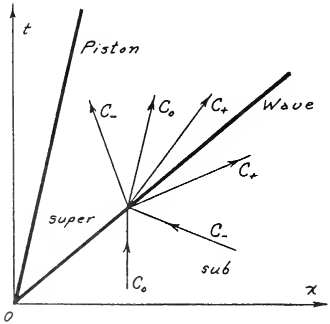
We distinguish four cases AA, AB, BA, BB according to whether the flow relative to the line W is supersonic (A) or subsonic (B) before or behind the front. In a schematic way, visualizing W and the characteristics as straight, these cases are represented by the following diagrams:



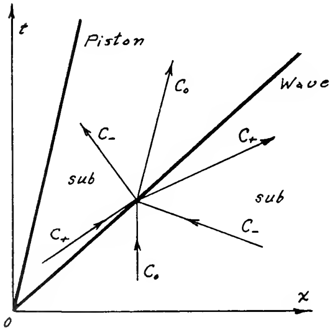
Case AA



Case AB



Case BA



Case BB

Figure 7
Possible relations between characteristics and reaction front.

This follows from the expressions for the directions $\frac{dx}{dt}$ along the characteristics. In the case of supersonic flow in front of the wave line, W is space-like observed from the region ahead of W; hence the initial values along the x-axis at $t = 0$ determine the flow uniquely up to W, no matter what the speed of the wave front is, as long as the flow ahead of W remains supersonic relative to W. Then by the transition conditions IV, V, VI, the values of p, u, ρ immediately behind the wave front are determined. Now there are, for any such wave velocity, two cases. In the first case, AA, also the flow behind W is supersonic relative to W. Then W is space-like for the region behind W, and in the domain between the wave front W and the piston curve the flow is uniquely determined by the additional boundary condition $u = U$ at the piston (assuming the piston curve time-like).

In the second case, AB, again the state ahead of W is uniquely determined by the initial conditions along $x \geq 0$ for $t = 0$. Then the transition conditions determine the state immediately behind W. Now, however, W is not space-like with respect to the zone behind W: Therefore the state immediately behind W, considered as initial state for the region between W and π, can no longer be prescribed, but has to be subjected to a restrictive condition to become compatible with the boundary condition at the piston curve π. While thus in the case AA the wave front W could be chosen arbitrarily, there is no leeway in the choice of W in the case AB. The situation is similar to that for non-reactive flow involving a <u>shock</u> line W; the velocity of the discontinuity front is determined in such a way that adjustment to the piston motion results.

In the case BA the wave line W observed from the state ahead of it is time-like, so that in front of W a boundary condition can be arbitrarily imposed. Consequently the wave speed and one quantity ahead of the wave could be prescribed: in other words, there are two degrees of indeterminacy in the case BA.

Thus, in addition to the initial state and the piston motion two more quantities can and must be prescribed to determine the flow.

In case BB, only one quantity such as the wave velocity can be prescribed; the other quantity in front of the wave must then be so chosen that the resulting values of the three quantities behind the wave are compatible with the boundary condition at the piston.

It should be emphasized that in both cases BA and BB the state ahead of the wave is influenced by the wave.

3. One-dimensional gas flow involving a reaction front.

Combined with Jouguet's rule, the preceding remarks now permit a complete discussion of our problem of determinacy for gas flow involving a detonation or deflagration front. The higher degree of indeterminacy of such flows as compared with flows involving merely shock discontinuities is due to the fact that all cases AA, AB, BA, BB may occur in reaction processes, while shocks always belong to the case AB. In the following detailed discussion we desist from a systematic enumeration of all mathematical possibilities of flows compatible with the conditions, and instead consider the most typical and important cases.

Since the state (0) ahead of the front is considered as given, the reaction wave i.e. its speed and the state (1) immediately behind it, is determined by one parameter for which we choose the velocity u behind the front. The speed of the wave then is given by $v_o = \frac{1}{\rho_o} \frac{p-p_o}{u}$. We first assume that W represents a detonation, with a state of rest, $u_o = 0$, ahead of it. Then as seen before we have $u > 0$. By $u_D = c_D$ we denote the particular value for the gas velocity behind the detonation front which satisfies the Chapman-Jouguet condition.

We start with the discussion of a weak detonation, which means $u < u_D$; according to Jouguet's rule this corresponds to

the cases AA, and W is space-like with respect to both adjacent regions. As a consequence, in view of the statements in the preceding section, every weak detonation, i.e., every speed for the detonation front compatible with the condition $u < u_D$ is mathematically compatible with the data. The state of the flow in front of W is determined by the initial condition, the state immediately behind W by the transition conditions, and it remains to analyze in detail how the adjustment of the velocity of the flow behind the front to the piston velocity is achieved for a fixed choice of our parameter u. This adjustment depends on whether or not the piston speed U exceeds the value u_D:

Case 1. $U < u_D$. In this case every weak detonation, u chosen arbitrarily in the interval $0 \leq u \leq u_D$, is possible.

We first consider detonations with $U \leq u \leq u_D$. Then the transition from the flow with $u > U$ behind the front to a flow with $u = U$ at the piston is effected by a centered rarefaction wave (see Fig. 8).

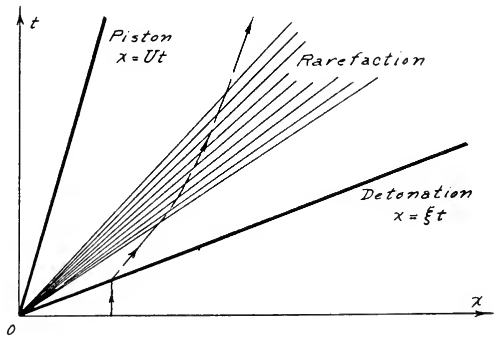


Figure 8
Weak detonation with $U < u < u_D$
for piston with $U < u_D$.

In the limiting Chapman-Jouguet case $u = u_D$, which separates weak and strong detonations, the rarefaction wave follows the detonation wave immediately (see Fig. 9):

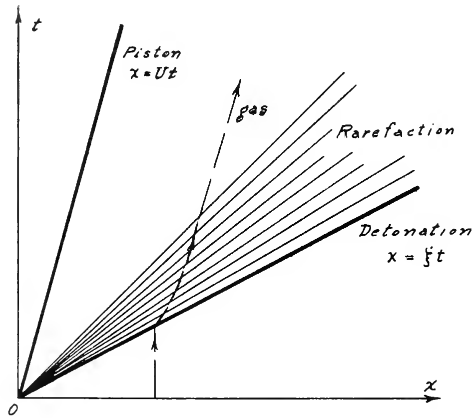


Figure 9
Detonation with $U < u = u_D$ satisfying Chapman-Jouguet's hypothesis.

In case $u = U$, the rarefaction wave drops out.
If the piston speed u exceeds u:

$$0 < u < U$$

the adjustment is effected by a shock (see Fig. 10).

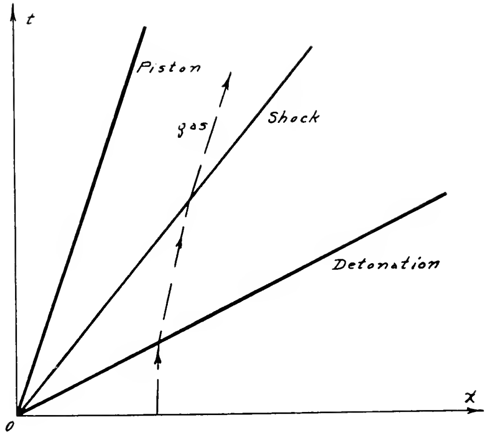


Figure 10
Detonation with $u < U$ for piston
with $u < u_D$.

Summing up: If $U < u_D$ a variety of weak detonations is compatible with the condition of the problem; specifically, any value of the velocity u behind the detonation front ranging from $u = 0$ (constant volume detonation with infinite detonation speed) to the value $u = u_D$ (Chapman-Jouguet) is possible.

It should be mentioned that there is an additional mathematical indeterminacy in the following sense: At any time after the start of the process the wave may change its character; then it may continue as a stronger (though still weak) detonation, followed by a rarefaction wave which will have to be balanced by a backward rarefaction wave,* or it continues as a weaker detonation wave, followed by a shock which is balanced by a backward shock (see Figs.11,12).

* As can be seen by the method described in the Manual [1],Chap.III, Section 48.

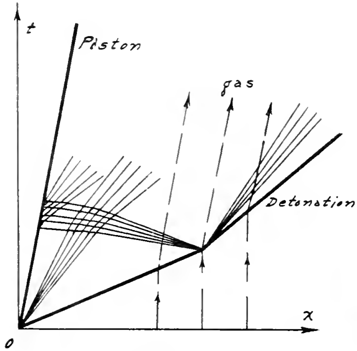


Figure 11
Detonation changing to
a weaker one.

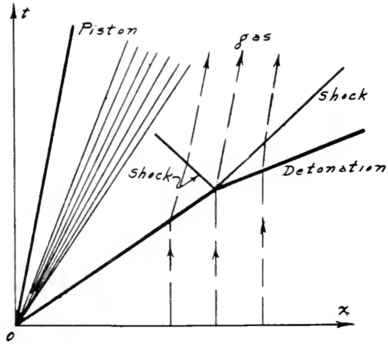


Figure 12
Detonation changing to
a stronger one.

Likewise, as appears from these last remarks, non-constant gradually changing detonation waves are compatible with the conditions.

Case 2. $U > u_D$. Also in this case a variety of weak detonations are possible; the adjustment to the piston velocity is then always effected by a shock. Again the range of these detonations can best be described by varying the parameter u from u = 0 (constant volume detonation) through positive values and by using a (p,u)-diagram.

It can be seen that there is a value $u = u_*$ of the parameter with $u_* < u_D$ for which the velocity $v_o = \frac{1}{\rho_o}\frac{p-p_o}{u^*}$ of the weak detonation is equal to the velocity of the shock which accelerates the gas to the piston speed $U > u_D$. Now every weak detonation with $u < u^*$ is possible followed by a (slower) shock which adjusts the particle velocity to the piston speed U. If the particle velocity u behind the weak detonation front is equal to u^* then the detonation front and the subsequent shock coalesce into one single discontinuity front W. Across this combined

front the laws of conservation are satisfied and the pressure is raised; hence the combined front, represented by W, is a detonation front. Since the velocity behind the combined front is $u = U > u_D$, this detonation is a strong detonation front, appearing as a weak detonation immediately followed by a shock.

While in the case $U > u_D$ there are again many weak detonations (with $u < u^*$) possible, there exists one well determined strong detonation, directly producing a constant flow of the same speeds as the piston (see Figs. 13 and 14). This is in agreement with the fact that, according to Jouguet's statement, a strong detonation corresponds to the case AB.

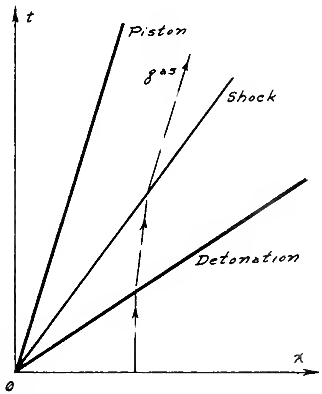


Figure 13
Detonation for piston with $u > u_D$.

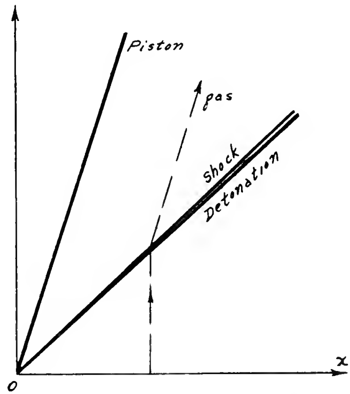


Figure 14
Detonation, for piston with $u > u_D$, immediately followed by a shock, both coalescing into a strong detonation.

As in Case I, the continuation of the wave is not uniquely determined; later changes, as explained in that case, are compatible with the conditions of the problem.

A strong detonation for $U < u_D$ is not compatible with the conditions; again we would have the case AB; for the domain behind the line W both W and the piston line π are time-like and along each of these lines u is constant; but no solution of the differential equations between two time-like lines can have such boundary values except the solution u = const; since $U < u_D$ at π and $u > u_D$ at W we have a contradiction, which shows that no adjustment of a strong constant detonation to a slow piston, e.g. a fixed wall, is possible.

As we have seen, however, a <u>Chapman-Jouguet detonation can always be adjusted to the piston motion</u>.

4. <u>Remarks on the Chapman-Jouguet hypothesis</u>.

Taking into account the physical situation, Jouguet has advanced the following argument in support of the Chapman-Jouguet condition.*

1. If the detonation is to proceed with a certain velocity, there are still two detonations possible, a weak one and a strong one, the latter equivalent to the weak detonation followed by a shock. The strong detonation leads, therefore, to the higher entropy of the burnt gas. Since the state with higher entropy is the more likely one, most probably the strong detonation will occur.

2. Due to heat conduction and rupture of the enclosure of the explosive, a rarefaction wave will be sent after the detonation wave.

3. Since the flow behind the strong detonation front is subsonic relative to the front, the rarefaction wave will catch up with the detonation and weaken it until the flow behind the front has become sonic, in other words, until the Chapman-Jouget condition is satisfied.

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* See Becker [6], p. 354.

Objection might be raised to the first step of the argument. Whether a weak detonation or a weak detonation immediately followed by a shock (i.e. a strong detonation) occurs, should depend on whether or not the resulting flow is compatible with the conditions.* These conditions are not explicitly stated by Jouguet and Becker, but a fixed closed end seems to be assumed. Then as seen before a strong detonation is not compatible with the conditions.

However, one may argue instead that the start of a detonation is a complicated process which cannot properly be described by gas flow involving a single discontinuity front. If such a description is appropriated for a somewhat later instant, adjustment of a <u>strong</u> detonation wave to a closed end could be achieved by a rarefaction wave sent back toward the closed end and being reflected there. This reflected wave would then overtake the detonation wave and weaken it until the Chapman-Jouguet condition is satisfied. (See Step 3 of Jouguet's argument).

Thus the second step of Jouguet's argument appears to be unnecessary. Although rarefaction waves due to rupture and heat loss will occur, a rarefaction wave is already needed for adjustment of the flow to the closed fixed end.

The final theoretical decision about the Chapman-Jouguet hypothesis is, of course, impossible without detailed investigation of the reaction process involved. Experimentally the Chapman-Jouguet hypothesis seems to be well confirmed. Therefore it appears reasonable to accept the hypothesis for calculations, even when non-steady processes are involved. Numerical procedure is facilitated because, as we saw, <u>the flow behind a constant Chapman-Jouguet detonation is a centered rarefaction</u>

* Just as the decision as to whether or not a shock occurs depends on whether or not the resulting flow is compatible with the conditions of the problem and not on the fact that the entropy increases through a shock.

wave, the first characteristic of which coincides with the detonation front. This fact, apparently not mentioned in the literature explicitly but implied in Taylor's work(see [14]), is decisive for actual calculation.

> If a detonation process is to be calculated for which the medium ahead of the wave is not in a constant state, and if the Chapman-Jouguet hypothesis is assumed to be valid, then one should be aware of the fact that the detonation front is no longer a characteristic of the flow behind it; it is rather an envelope of (not necessarily straight) characteristics (see Figs. 15,16).

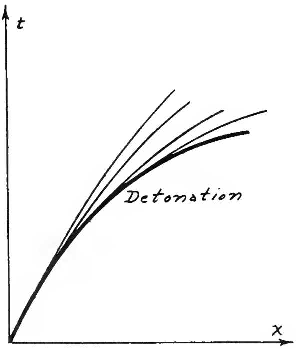

Figure 15
Accelerating Chapman-Jouguet detonation wave.

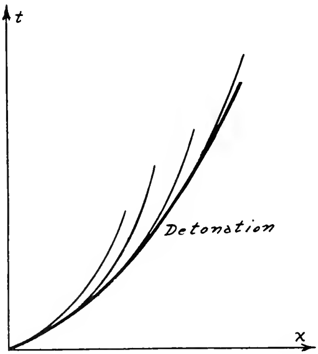

Figure 16
Decelerating Chapman-Jouguet detonation wave.

(in specific cases it was found by computation that the deviation of the detonation front from a single characteristic is very slight.)

5. In <u>deflagration</u> processes the situation is in many respects quite different from that encountered with detonation processes. Suppose a weak deflagration wave (practically all observed deflagration waves seem to be weak) begin at the piston, $x = 0$, and moves into the unburnt gas in the interior of the tube, $x = 0$. Then the velocity u of the burnt gas behind the deflagration front is negative, $u < 0$. This is compatible with the conditions of the problem only if the piston is moved out with a speed at least equal to u. Otherwise it is impossible that a deflagration front moves into the explosive gas. For, we have here the case BB; the line W is time-like with respect to the region behind it; an adjustment to a forced or forward moving piston is impossible. What one observes actually, is that <u>a pre-compression wave is sent out into the explosive</u>. It pushes the explosive ahead with a velocity just sufficient to insure that it may come to rest when it is swept over by the deflagration wave.

The occurrence of a pre-compression wave is in complete agreement with, or rather, a consequence of, the theoretical considerations of Section 2. Indeed, according to Jouguet's rule, the flow ahead of a deflagration is subsonic (case B) and consequently the deflagration influences the state of the gas ahead of it. Observation* shows that the deflagration wave moves with gradually increasing velocity and sends out a continuous compression wave into the explosive. A simplified model, however, is more convenient for theoretical treatment: to assume that a single constant shock wave is sent out into the explosive and that the deflagration wave follows with a constant velocity (see Figs. 17 and 18).

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* See Payman [9], [10], [11], [12], [13].

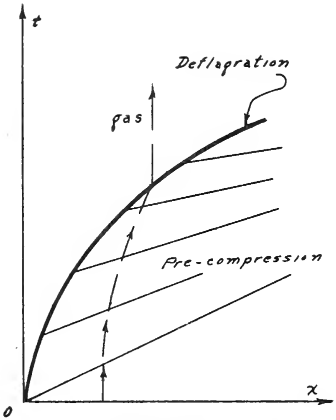

Figure 17
Actual deflagration with continuous pre-compression, $U = 0$.

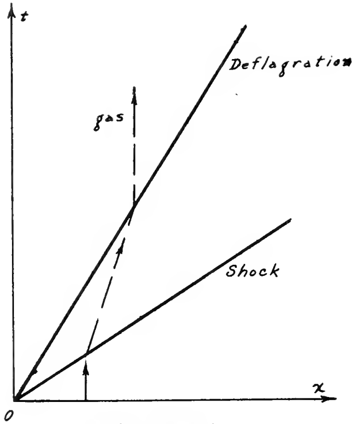

Figure 18
Deflagration with shock pre-compression, $U = 0$.

To each assumed pre-compression shock there is still a great variety of possible adjustments either by a uniquely determined weak deflagration wave (case BB) or by a strong deflagration wave (case BA) which may be arbitrarily chosen within a one parameter variety, and is followed again by rarefaction or shock waves (see Figs, 19 and 20).

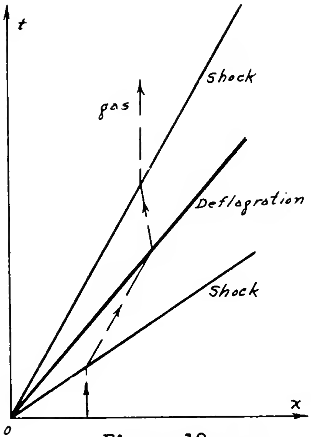

Figure 19
Deflagration followed
by shock.

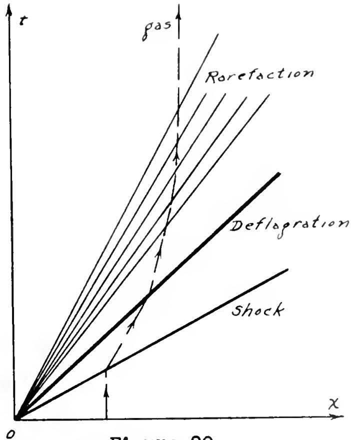

Figure 20
Deflagration followed
by rarefaction.

These possibilities are, of course, limited by the condition that the velocity of each wave is less than that of the preceding wave. If the shock wave following a strong deflagration wave coalesces with it the result is equivalent to a weak deflagration wave. A deflagration wave, however, coalescing with its pre-compression wave is equivalent to a detonation wave.* Thus, by increasing the strength of the pre-compression wave so much that its speed equals that of

* Cf. Jouguet [5], v. Neumann [7].

a possible detonation, one may envision a continuous transition from deflagration processes to detonation processes.

The main point in this discussion is that deflagration processes have a higher degree of indeterminacy than detonation processes. The Chapman-Jouguet condition does not make the process unique; for sufficiently strong arbitrary pre-compression shocks a rarefaction wave following the Chapman-Jouguet deflagration can always be found such that the flow at the piston has the right velocity. To determine the process, new physical conditions will have to be added that must arise from a study of combustion processes.

It is clear that there are still more possibilities in addition to those mentioned, viz. those in which the pre-compression is not achieved by a shock, but by a continuous compression wave. As was mentioned before, this is what actually happens when deflagration begins at one end of the tube. The Chapman-Jouguet condition can then not be satisfied in the first stage of the process, but only one additional condition is needed as long as the deflagration is weak. When the deflagration becomes strong a second condition (such as the Chapman-Jouguet condition) must be added.

6. Stationary deflagration fronts.

So far we have mainly been concerned with reaction waves moving in to an explosive gas at rest. It is naturally of great interest to study other situations such as flows involving a stationary reaction front.

Let us consider in particular the case that a steady flow of combustible gas enters one end of a cylindrical tube, undergoes combustion on passing through a deflagration or "flame" front, and escapes as burnt gas at the other end of the tube. (In a crude approximation the flow through the combustion chamber of a jet engine might be described in such a fashion.)

Suppose the velocity of the entering explosive gas is varied, the state of the explosive being kept fixed; then, if at all, two deflagration processes are possible, a weak and a strong one. We may assume the weak one occurs. If the entrance velocity is increased, the pressure drop across the deflagration front will increase until for a certain value of the entrance speed v_C the Chapman-Jouguet reaction is reached. A further increase in pressure drop would make the deflagration a strong one and at the same time reduce the entrance velocity; for, by property Q (see p. 9), the relative speed between explosive and reaction front is a maximum for a Chapman-Jouguet deflagration. If, therefore, the entrance speed is increased beyond v_C, a stationary deflagration process is no longer possible, and the combustion will therefore be stopped. This phenomenon has been termed "choking." Its occurrence is another illustration for the importance of the Chapman-Jouguet condition.

7. Remark on Taylor's Theory of Spherical Detonation Waves.

Taylor has shown that the flow produced by a spherical detonation wave is accessible to mathematical treatment. The detonation is assumed to be weak or to be a Chapman-Jouguet detonation. It is then not influenced by the flow behind it. This was shown earlier to be true for one-dimensional waves and the fact remains true for spherical waves. Therefore, it is consistent to assume that the detonation front proceeds with constant radial speed. In this respect a spherical detonation wave differs from a spherical blast wave, which is influenced by the flow behind it and gradually fades out unless energy is constantly added.

For the flow behind the wave it is assumed that velocity u, pressure p, and specific volume τ depend only on the ratio $\xi = r/t$, r being the distance from the center and t the time. This assumption is consistent with a constant speed $\xi = \Xi$ of the detonation front. The differential equations for spherical flow

$$(8) \qquad u_t + uu_r + tp_r = 0, \ \tau_t + u\tau_r - (u_r + 2r^{-1}u)\tau = 0$$

together with $p^{-1}dp + \tau^{-1}d\tau = 0$ (see Manual [1], V 90,p.263, (34), (35)), then become

$$(9) \qquad (u - \xi)u_\xi - \gamma p \tau = 0 \qquad \text{and}$$

$$(u - \xi)\tau - (u_\xi + 2\xi^{-1}u)\tau = 0 \ ,$$

(cf. Manual V90, p. 265, (34)', (35)'). Elimination of τ_ξ and introduction of $c^2 = \gamma p \tau$ yields the equation

$$(10) \qquad u_\xi = 2\xi^{-1}u\left[\left(\frac{\xi - u}{c}\right)^2 - 1\right]^{-1} .$$

This equation was solved numerically under the conditions that behind the detonation front $r = \Xi t$, the quantities u, p, τ assume the values which are uniquely determined by the transition conditions of the reaction process. If this process satisfies the Chapman-Jouguet condition, the rate of change of velocity with respect to the variable ξ is infinite behind the front (while it is finite for one-dimensional reactions). This is seen from (10) since condition S (p. 9) implies that $\Xi - u = |v| = c$. If the detonation is weak, the rate of change of velocity is finite but different from zero in contrast to one-dimensional flow behind a weak detonation wave, which implies a section on which the state is constant (see Fig. 8).

However, the important feature of the solution is that for a certain positive value of r/t, say $r/t = \xi_0$, the velocity is zero. The path $r = \xi_0 t$ is a characteristic which separates the rarefaction wave between $r/t = \xi_0$ and $r/t = \Xi$ from a constant state between $r/t = 0$ and $r/t = \xi_0$.

Thus, a spherical flow involving a detonation wave with constant speed and an expanding core in which the state is constant and the gas at rest, is found to be consistent with the general laws of detonation.

By determining such a flow, Taylor has disproved the statement made by Jouguet, quoted on occasion, that a spherical detonation wave could not maintain itself.

Appendix I

In this Appendix we present justifications for some statements made in the text.

I.1. We first prove the equivalence of the various forms (P) to (T) of the Chapman-Jouguet condition:

From (1) and (2) we have

$$(\gamma - 1)(dE + pd\tau) = \tau(dp + \gamma\tau^{-1}pd\tau) = \tau(dp + \rho^2c^2d\tau);$$

from IV and (4) we have

$$2(dE + pd\tau) = (\tau_0 - \tau)dp + (p - p_0)d\tau$$

$$= (\tau_0 - \tau)\ (dp + \rho^2v^2d\tau)$$

$$= (\tau_0 - \tau)^2\rho_0^2dv_0^2 \quad .$$

Vanishing of any of these terms is therefore equivalent. Relation $dE + pd\tau = 0$ expresses that the entropy is stationary (R); relation $dp/d\tau = -\gamma p/\tau$ is condition (T); relation $dp/d\tau = -(p-p_0)/(\tau_0 - \tau)$ is condition (P); relation $dv_0^2 = 0$ is condition Q. Vanishing of $dp + \rho^2c^2d\tau$ and $dp + \rho^2v^2d\tau$ is equivalent with $|v| = 0$ or condition (S).

I.2. To establish that the speed $|v_0|$ and the entropy have a minimum and a maximum in the Chapman-Jouguet states D and C, we should show that the derivatives of $\frac{dv_0^2}{d\tau}$ and of $\theta^{-1}(\frac{dE}{d\tau} + p)$ with respect to τ are positive in state D and negative in state C. Since these expressions vanish at these points and $\tau_0 - \tau$ is positive in D, negative in C, it is sufficient to

show that the derivative of

$$(\tau_o - \tau)\rho_o^2 dv_o^2/d\tau = 2(\tau_o-\tau)^{-1}\left(\frac{dE}{d\tau} + p\right) = \frac{dp}{d\tau} + \left(\frac{p-p_o}{\tau_o - \tau}\right)$$

is positive in D and C. Since $d\left(\frac{p-p_o}{\tau_o-\tau}\right) = 0$ in D and C, the statement follows from the assumption that $d^2p/d\tau^2 > 0$ (see p.10).

I.3 In order to justify Jouguet's rule we make an additional assumption on the nature of the burnt and unburnt gases: Through a detonation the pressure rises more than it would through adiabatic compression to the same volume, and through deflagration the pressure drops less than through adiabatic expansion to the same volume. Let p_o, τ_o, p_1, τ_1 be pressure and specific volume ahead of and behind the reaction front and let p_a be the value of the pressure by adiabatic change from the state (p_o, τ_o) to a state with $\tau = \tau_1$. Then the assumption above is simply

$$p_a < p_1$$

for detonation and deflagration.

We further employ for the unburnt gas the standard assumption (see Manual I-2(a), p.6)

$$\frac{d^2p}{d\tau^2} > 0 \quad \text{for adiabatic change}$$

observing that the sound velocity c_o is given by

$$c_o^2 = -\tau_o^2\left(\frac{dp}{d\tau}\right)_o$$

we have as a consequence of the latter assumption, depending on whether $\tau_1 < \tau_o$ or $\tau_1 > \tau_o$,

$$c_o^2 \lessgtr \tau_o^2 \frac{p_a - p_o}{\tau_o - \tau_1}$$

and by virtue of the assumption $p_a < p_1$,

$$c_o^2 \lesseqgtr \tau_o^2 \frac{p_1 - p_o}{\tau_o - \tau_1},$$

whence by (4),

$$c_o^2 \lesseqgtr v_o^2 \quad .^*$$

Since $\tau_1 < \tau_o$ for detonations, $\tau_1 > \tau_o$ for deflagrations we have proved that <u>the flow ahead of the reaction front</u> and observed from it is <u>supersonic for detonations and subsonic for deflagration</u>.

To prove <u>Jouguet's rule for the state behind the front</u> we first obtain from the relations enumerated in App. I.1 the expression

$$\frac{v^2}{c^2} = 1 + \frac{1}{(1+\mu^2)} [\tau - \mu^2 \tau_o] \left(\frac{dp}{d\tau} + \frac{p - p_o}{\tau_o - \tau}\right) .$$

The derivative with respect to τ for the Chapman-Jouguet states D and C,

$$\frac{d}{d\tau}\left(\frac{v^2}{c^2}\right) = \frac{1}{(1+\mu^2)p} [\tau - \mu^2 \tau_o] \frac{d^2p}{d\tau^2} = - \frac{p + \mu^2 p_o}{(1+\mu^2)p} \frac{d^2p/d\tau^2}{dp/d\tau} > 0$$

according to the assumptions made earlier (p. 10). The Mach number $M = v/c$ equals 1 in the states D and C by condition S, and these are the only states satisfying the Chapman-Jouguet condition. Therefore $M > 1$ if $\tau < \tau_D$ or $\tau < \tau_C$: in view of $dp/d\tau < 0$ this is equivalent to $p < p_d$, $p < p_c$; the former case corresponds to a weak detonation, the latter to a strong deflagration (see p. 15). Similarly $M < 1$ for $p < p_D$ and $p < p_C$.

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* This argument is similar to that for shocks in the Manual, see III-32, p. 80. If, instead of the assumption $p_a < p_1$, one assumes that the sound velocity increases through a constant volume detonation, the argument of H. Weyl([15] p. 10) could also be used.

Appendix II

Outline of a procedure to compute a Chapman-Jouguet reaction if the state in front of it is given.

We shall indicate briefly how to determine the Chapman-Jouguet reactions when the state of the explosive is given. The given quantities are pressure p_o, specific volume τ_o, the quantity θ_o, which is proportional to the temperature, and the total energy per unit mass, E_o. Further, the function $E = E(\theta)$ (and consequently $\gamma = \gamma(\theta) = 1 + d\theta/dE$) are assumed to be known. The problem is then to determine p, γ, θ, E on the basis of the relations I, II, IV and one of the conditions (P) to (T).

We proceed to eliminate v_o, v, τ, and β obtaining one equation for the quantity θ, which then is to be solved.

Eliminating v_o from the relations I and II, setting $v = c$ by condition S and $c^2 = \gamma\theta$ by (2) we obtain

$$p - p_o = \rho\left(\frac{\rho}{\rho_o} - 1\right)\gamma\theta .$$

Substituting $\tau = \theta/p$ and $\tau_o = \theta_o/p_o$ in the latter equation and in IV we obtain

$$1 - \frac{p_o}{p} = \gamma\left(\frac{\theta_o}{\theta}\frac{p}{p_o} - 1\right)$$

and

$$\frac{E-E_o}{\theta} = -\frac{1}{2}\left(1 - \frac{\theta_o}{\theta}\frac{p}{p_o}\right)\left(1 + \frac{p_o}{p}\right) .$$

Our intention is to eliminate p_o/p ; before doing this we introduce the dimentionless quantities

$$\xi = \frac{p_o}{p}, \quad \eta = \frac{\theta_o}{\theta}, \quad \zeta = \frac{E-E_o}{\theta} .$$

Then the two equations above become

$$2\gamma\zeta = 1 - \xi^2$$

$$\gamma(\eta - \xi) = \xi - \xi^2 .$$

Eliminating ξ we obtain

$$\zeta = \frac{1}{4\gamma}\left\{\pm(\gamma+1)\sqrt{(\gamma+1)^2 - 4\gamma\eta} - (\gamma+1)^2 + 2(1+\gamma\eta)\right\}$$

$$= \frac{2+\gamma-2\eta-\gamma\eta^2}{\pm(\gamma+1)\sqrt{(\gamma+1)^2-4\gamma\eta} + (\gamma+1)^2 - 2(1+\gamma\eta)} ;$$

the upper sign corresponds to detonations, the lower sign to deflagrations.

Suppose Θ_0 and E_0 are given, then η, γ, and ζ depend on Θ only. The latter equation is therefore an equation in Θ only, which can easily be solved by iterations. For detonations, for which η is rather small, the equation can in good approximation be replaced by

$$\zeta = \frac{1}{2\gamma}\left(1 - \frac{\gamma^2}{(\gamma+1)^2}\eta^2\right) , \text{ or}$$

$$E = E_0 + \frac{\Theta_0}{2(\gamma+1)}\left(\frac{\gamma+1}{\gamma}\frac{\Theta}{\Theta_0} - \frac{\gamma}{\gamma+1}\frac{\Theta_0}{\Theta}\right) .$$

Once ζ is found one obtains ξ from $p_0/p = \xi = (1+\gamma\eta-2\gamma\zeta)/(\gamma+1)$, further $\tau_0/\tau = \eta/\xi$. Finally one obtains v_0 and v_1 from (4). Thus all quantities characterizing a Chapman-Jouguet reaction are determined.

Appendix III

Von Neumann's argument in support of the Chapman-Jouguet condition

A significant argument in order to justify the Chapman-Jouguet condition was given by von Neumann [16]. It is essentially based on the fact (disregarded so far in the present report) that the reaction actually does not take place in a sharp discontinuity front but rather in a zone of finite width. In the terms employed before, the result of this argument can be formulated as follows: Strong deflagrations and weak detonations do not exist provided that the reaction process is strictly one-dimensional and stationary when observed from the moving reaction zone, and provided that a certain additional condition is satisfied.

To see this let us consider an explosive confined to a tube extended along the x-axis. The unexploded gas contained in the space $x < 0$ and being in the state (0) moves with the velocity v_0 into the cross-section $x = 0$ at which a deflagration reaction is to start. While in the preceding parts of this report the transition from the state (0) to the completely burnt state (1) was considered a discontinuous process, we now turn to a more refined discussion visualizing the transition as continuous, gradually leading from the state (0) through intermediate states in which the gas is incompletely burnt, to the final state (1). The reaction zone may extend from the cross-section $x = 0$ to the cross-section $x = a$ out of which the completely burnt gas moves with the velocity v_1 into the space $x > a$. At each cross-section of the reaction zone, where, as assumed, all quantities remain stationary, the gas

is a mixture of a certain fraction, n, of burnt gas and a fraction, 1 - n, of still unburnt gas. We assume that the fraction n increases monotonically on passing across the reaction zone from the value $n = 0$ at $x = 0$ to the value $n = 1$ at $x = a$. The value of n then completely characterizes the cross-section in the reaction zone.

It is an important point of the argument, originally suggested by G. I. Taylor, that each section of the reaction zone between the unburnt state (0) and the intermediate state (n) is subject to the same laws of conservation which so far were analyzed in Section 1 for a complete reaction process. The mixture of burnt and unburnt gases characterized by the fraction n possesses a certain energy function $E = E_n(\Theta)$ and therefore the same conclusions concerning possible transitions can be drawn as were drawn in Section 1 for a completed reaction. Therefore, a relation between the possible values of pressure p and specific volume τ in the state (n) can be established, and accordingly we may draw a family of transition curves in a (τ, p)-plane as in Figure 21, each member corresponding to a value of n between 0 and 1. The curve for $n = 0$ is the shock transition curve; it passes through the point (τ_0, p_0). The curve for $n = 1$ contains the values (τ_1, p_1) characterizing the state (1). We now introduce the additional condition, announced before: <u>the family of transition curves has no envelope</u> and behaves as indicated in the diagram Figure 21.

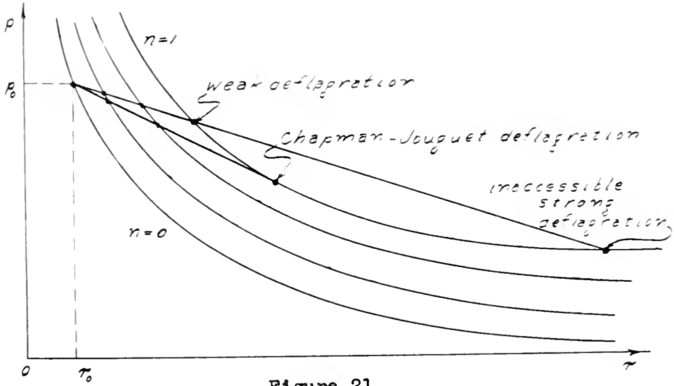

Figure 21

Family of transition curves in the (τ,p)-plane for incomplete deflagrations, fractions, n, of completion ranging from n = 0 to n = 1.

Since for all these transitions the initial state (τ_0,p_0) and the initial velocity v_0 are the same, formula

$$\frac{p-p_0}{\tau_0-\tau} = \rho_0^2 v_0^2 \tag{4}$$

is applicable; it states that the points (τ,p) for all intermediate transitions lie on a fixed straight line, Λ. The points of intersection of this straight line with the transition curves (n) determine the actual partial transition from (0) to (n). On the line Λ we follow the set of points of intersection, letting n range from 0 to 1; thus we obtain a well-defined point of intersection on the curve n = 1. (See

Figure 21). By virtue of the assumption that the family of transition curves behaves as is indicated in the diagram, Figure 21, no other point of intersection of the line Λ with the curve $n = 1$ can be reached through intermediate points on the line Λ, since this would imply passing through a section of this line not corresponding to partial transitions for $n < 1$. Now it is obvious that the points on the transition curves $n = 1$ that can be reached in this way correspond to weak or to Chapman-Jouguet deflagrations. Thus strong deflagrations are excluded.

This result is sufficient to exclude also the possibilities of weak detonations under the assumptions made. As was mentioned in Section 5, p. 29, a detonation process can be considered a deflagration process coalescing with a pre-compression shock preceding it. It is immediately seen that, in particular, a weak detonation process corresponds to a shock followed by a strong deflagration process. Exclusion of strong deflagrations therefore implies exclusion of weak detonations.

Inasmuch as strong detonations, under proper boundary conditions, are also excluded, the Chapman-Jouguet detonations remain as the only possible ones.

In conclusion it should, however, be said that the assumptions of one-dimensional, i.e. of strictly confined flow, is never completely satisfied in reality for detonation processes and that the possibility of three-dimensional weak detonations is therefore not completely excluded by the reasoning presented.

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